SOLUTION OF NONLINEAR HEAT-CONDUCTION PROBLEMS IN A MEDIUM WITH PHASE TRANSITIONS

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A new method is proposed for solving one-dimensional nonlinear heat-conduction problems in a medium having any number of phase transitions (nonlinear Stefan problems). The method consists of a direct calculation of the isotherms and reduces to the Cauchy problem for a system of ordinary differential equations.

1. Formulation of the problem. The heat-conduction equation is

$$\frac{\partial E(T)}{\partial t} = \operatorname{div}[k(T)\operatorname{grad} T] \quad (k(T) > 0), \tag{1.1}$$

where t is the time, T is the temperature, E is the internal energy of the medium, and k is the thermal conductivity coefficient. The function E(T) is monotonic (nondecreasing) and has a discontinuity of the first kind at the T values at which the medium changes phase. The discontinuity in E(T) at these points is equal to the heat of the phase transition (e.g., to the heat of fusion at the melting point). The functions E(T) and k(T) are assumed known.

Using the change of variables (T_0 is an arbitrary constant)

$$\int_{T_0}^{T} k(z)\,dz = u,$$

we convert Eq. (1.1) to

$$\frac{\partial f(u)}{\partial t} = \Delta u, \quad f(u) = E(T(u)). \tag{1.2}$$

Here, evidently, u is a monotonic function of T and $f(u)$ is a monotonic (nondecreasing) function of u which may have discontinuities of the first kind. Figure 1 shows a typical plot of the $f(u)$ dependence. For the case of one-dimensional heat conduction along the x-axis, Eq. (1.2) becomes

$$\frac{\partial f(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

The problem of determining the temperature of the medium reduces to that of finding the solution u(x, t) of Eq. (1.3) for arbitrary initial and boundary conditions. The derivatives of the function u have discontinuities at those curves in the x, t plane at which u takes on a value at which a phase transition occurs [i.e., one of the u values corresponding to the discontinuities of the function $f(u)$]. These curves in the x, t plane trace out the motion of the phase-transition fronts. This problem (the nonlinear Stefan problem) has been the subject of many studies (see [1] for a bibliography).

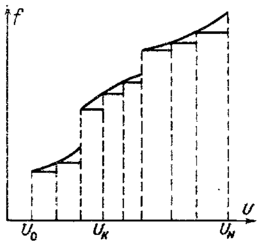

Fig. 1

2. Isotherms. Let us describe one approach to a numerical solution of this problem. In the region in which the function $f(u)$ is defined, we choose points u_k with $k = 0.1, \ldots, N$ such that $u_k < u_{k+1}$ for all k and such that the distance between neighboring points is fairly small (in particular, these distances may be equal). Among the points u_k are all the discontinuities of the function $f(u)$. We introduce the discontinuous piecewise-constant function F(u) by means of

$$F(u) = F_k = f(u_k) \quad \text{for} \quad u_k \leqslant u < u_{k+1} \quad (k = 0, 1, \ldots, N-1). \tag{2.1}$$

Figure 1 illustrates this function, which approximates the function $f(u)$. We replace Eq. (1.3) by

$$\frac{\partial F(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{2.2}$$

which we solve below. At those points of the x, t plane at which u is not equal to any of the u_k, the function F(u) is constant, according to Eq. (2.1). Then it follows from Eq. (2.2) that $\partial^2 u/\partial x^2 = 0$; i.e., u is a linear function of x. For each fixed value of t, therefore, the solution u(x, t) of Eq. (2.2) will be a piecewise-linear function of x with breaks (discontinuities in the derivative $\partial u/\partial x$) at $u = u_k$, $k = 0, 1, \ldots, N$. To determine the solution u(x, t), therefore, it is sufficient to find the $x = x_k(t)$ curves in the x, t plane at which $u = u_k$, $k = 0, 1, \ldots, N$. These curves, at which the derivative $\partial u/\partial x$ has a discontinuity, are evidently isotherms.

We introduce the differential equations for the isotherms. We assume that the isotherm $x_k(t)$, at which $u = u_k$, lies at some t between the isotherms $x^-(t)$ and $x^+(t)$ corresponding to $u = u^-$ and $u = u^+$, i.e., $x^- < x_k < x^+$. Here u^- and u^+ are the discontinuities of the function F(u) adjacent to u_k; They may be equal to u_{k-1}, u_k, or u_{k+1}. Since u(x, t) is a piecewise-linear function of x and F(u) is a piecewise-constant function of u, we have

$$\begin{aligned} u &= u_k + \frac{u^- - u_k}{x^- - x_k}(x - x_k), \; F = F^- \quad (x^- < x < x_k), \\ u &= u_k + \frac{u^+ - u_k}{x^+ - x_k}(x - x_k), \; F = F^+ \quad (x_k < x < x^+). \end{aligned} \tag{2.3}$$

Here F^- and F^+ are the constant values taken on by the function F(u) in (2.1) in the corresponding u ranges.

We now let a and b be fixed values of x lying at some t in the intervals (Fig. 2)

$$x^- < a < x_k, \qquad x_k < b < x^+.$$

Integrating both parts of Eq. (2.2) over x from a to b at fixed t, we obtain

$$\int_a^b \frac{\partial F(u)}{\partial t} dx = \frac{\partial u}{\partial x}(b, t) - \frac{\partial u}{\partial x}(a, t).$$

Since a and b are constants, the differentiation with respect to t and the integration over x on the left-hand side of this equation can be interchanged; then the integral is easily calculated with the help of Eq. (2.3). Also using Eq. (2.3) to transform the right-hand side of this equation, we find

$$\frac{d}{dt}[(x_k - a) F_{\text{,}} + (b - x_k) F^+] = \frac{u^+ - u_k}{x^+ - x_k} - \frac{u^- - u_k}{x^- - x_k}.$$

Differentiating with respect to t, we find

$$\frac{dx_k}{dt}(F^- - F^+) = \frac{u^+ - u_k}{x^+ - x_k} - \frac{u^- - u_k}{x^- - x_k}. \tag{2.4}$$

Let us consider the various particular cases. We assume that near $x = x_k$ the function u(x, t) either monotonically increases or monotonically decreases with increasing x. In these cases we have

$$\begin{aligned} &x^- = x_{k-1}, \; u^- = u_{k-1}, \; x^+ = x_{k+1}, \; u^+ = u_{k+1}, \; F^- = F_{k-1}, \; F^+ = F_k; \\ &x^- = x_{k+1}, \; u^- = u_{k+1}, \; x^+ = x_{k-1}, \; u^+ = u_{k-1}, \; F^- = F_k, \; F^+ = F_{k-1}. \end{aligned}$$

In both cases, Eq. (2.4) becomes

$$\frac{dx_k}{dt} = \frac{1}{F_{k-1} - F_k}\left(\frac{u_{k+1} - u_k}{x_{k+1} - x_k} - \frac{u_{k-1} - u_k}{x_{k-1} - x_k}\right). \tag{2.5}$$

We assume both isotherms adjacent to x_k correspond to values $u = u_{k-1}$, less than u_k. Then in Eq. (2.3) and (2.4) we must set

$$x^- = x_{k-1}, \quad x^+ = x'_{k-1}, \quad u^- = u^+ = u_{k-1}, \quad F^- = F^+ = F_{k-1}.$$

In this case, the left-hand side of Eq. (2.4) vanishes, and the equation itself leads to a contradiction, yielding the equality $u_k = u_{k-1}$. In this case the $x = x_k$ isotherm cannot exist: it "ends" instantaneously, and $u(x_k, t)$ is made equal to u_{k-1}. Analogously, isotherms $x_k(t)$ cannot exist such that both neighboring isotherms correspond to values $u = u_{k+1} > u_k$.

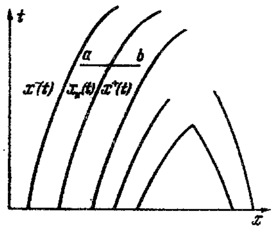

Fig. 2

We assume that two of the three neighboring isotherms, $x_k(t)$ and $x_k(t)$, correspond to the same value $u = u_k$, while the third, $x_{k-1}(t)$, corresponds to a smaller value $u = u_{k-1}$. We assume for definiteness that the last of these isotherms lies to the left of the first two; i.e., we set

$$x^- = x_{k-1}, \; x^+ = x'_k, \; u^- = u_{k-1}, \; u^+ = u_k, \; F^- = F_{k-1}, \; F^+ = F_k$$

in Eqs. (2.3) and (2.4). Then Eq. (2.4) becomes

$$\frac{dx_k}{dt} = \frac{u_k - u_{k-1}}{(F_k - F_{k-1})(x_k - x_{k-1})} > 0\,. \tag{2.6}$$

In this case the isotherm $x_k(t)$ always moves to the right in the x, t plane, in accordance with (2.6). In contrast with Eq. (2.5), the right-hand side of Eq. (2.6) depends only on the neighboring isotherm x_{k-1} on the left, and does not depend on the neighboring isotherm x'_k on the right. In this case, the $x_k(t)$ isotherm does not "sense" what occurs on its right.

We assume four neighboring isotherms x_{k-1}, x_k, x_k', x_{k-1}' are arranged in such a manner that $x_{k-1} < x_k < x_k' < x_{k-1}'$; these isotherms correspond to $u_{k-1}, u_k, u_k, u_{k-1}$, respectively. Equation (2.6) is then valid for the isotherm $x_k(t)$, and an analogous equation will hold for the isotherm $x'_k(t)$:

$$\frac{dx_k'}{dt} = \frac{u_k - u_{k-1}}{(F_k - F_{k-1})(x_k' - x_{k-1}')} < 0\,. \tag{2.7}$$

We see from Eqs. (2.6) and (2.7) that the isotherms $x_k(t)$ and $x'_k(t)$ do not affect each other; they approach each other. When they intersect, the interval $[x_k, x'_k]$ of the x-axis on which $u = u_k$ contracts to a point. Then both isotherms end, and the interval of the x-axis on which $u = u_k$ ceases to exist. The merging of the isotherms is shown schematically at the right in Fig. 2. Physically, this phenomenon has a very simple meaning: if there is an elevated-temperature region ($u = u_k$) on both of whose sides the temperature is lower, heat conduction will cause this region to contract and ultimately disappear.

There is another case to consider—that in which two of three neighboring isotherms correspond to the same $u = u_k$ value, while the third, either to the left or to the right of the first two, corresponds to a greater $u = u_{k-1}$. This case would be completely analogous to that above if the piecewise-function F(u) in (2.1) were continuous on the left but not on the right. With this choice of F(u), it turns out that the central isotherm of the three ends instantaneously, as in one of the cases considered previously.

3. Boundary and initial conditions. Equations (2.4)–(2.7) hold for interior isotherms $x_k(t)$, i.e., those between two other isotherms. To derive equations for the boundary isotherms, we must appeal to the boundary conditions. We assume that the solution of problem (1.3) or (2.2) is to be sought in the region of the x, t plane bounded by the inequalities $g_1(t) \le x \le g_2(t)$, $t \ge 0$, where g_1 and g_2 are specified functions.

We assume that a boundary condition of one of the three following types is specified on the $x = g_1(t)$ curve:

$$u = h_1(t), \quad \partial u/\partial x = h_2(t), \quad \partial u/\partial x = h_3(u, t). \tag{3.1}$$

Let us consider the isotherm $x_k(t)$ corresponding to $u = u_k$. Let us assume that at some t it lies between the boundary $x = g_1(t)$ and the isotherm $x_{k+1}(t)$; i. e., that $g_1 < x_k < x_{k+1}$. We assume that v and q are the values of u and $\partial u/\partial x$ at the boundary $x = g_1(t)$; here $u_{k-1} < v < u_k$. Then the equation analogous to (2.4) is

$$\frac{dx_k}{dt} = \frac{1}{F_{k-1} - F_k}\left(\frac{u_{k+1} - u_k}{x_{k+1} - x_k} - q\right). \tag{3.2}$$

Here q is determined from the following equalities corresponding to the three boundary conditions (3.1):

$$q = \frac{u_k - h_1(t)}{x_k - g_1(t)}, \quad q = h_2(t), \quad q = h_3[u_k + (g_1(t) - x_k)\,q, \; t]. \tag{3.3}$$

In the first two cases, q is specified by the explicit equation (3.3), while in the third case a transcendental equation must be solved to determine q. Substituting q from (3.3) into (3.2), we find the desired equation for the boundary isotherm $x_k(t)$. This equation will hold until the following inequalities become valid:

$$u_{k-1} < v = u(g_1(t_1), t) = u_k + (g_1(t) - x_k)\,q < u_k. \tag{3.4}$$

If at some time t_0 the left inequality in (3.4) is violated (i. e., if $v = u_{k-1}$ at $t = t_0$), then a new isotherm $x_{k-1}(t)$ leaves the boundary $x = g_1(t)$ at this time. This isotherm will be the boundary isotherm at $t > t_0$, while the isotherm $x = x_k(t)$ will become an interior one. For the new isotherm $x_{k-1}(t)$, we have the obvious initial condition $x_{k-1}(t_0) = g_1(t_0)$. If, on the other hand, the right inequality in (3.4) is violated at $t = t_0$, we have $v = u_k$ and $x_k = g_1$ at $t = t_0$. In this case, the isotherm $x_k(t)$ moves to the boundary and disappears, while the interior isotherm $x_{k+1}(t)$ becomes the boundary isotherm at $t > t_0$.

We assume that at time $t = t_0$ the isotherm $x_k(t)$ intersects the boundary $x = g_1(t)$, at which the first of boundary conditions (3.1) is specified. Then $u_k = h_1$ and $x_k = g_1$ at $t = t_0$, so the first of equations (3.3) for q becomes indeterminant. Equation (3.2) has a singularity at $t = t_0$. Applying l'Hôpital's rule to the indeterminant form for q, we find

$$q(t_0) = \frac{dh_1/dt}{dg_1/dt - dx_k/dt} \quad \text{for} \quad t = t_0. \tag{3.5}$$

Substituting (3.5) for q into Eq. (3.2) and setting $t = t_0$ in it, we find an algebraic (quadratic) equation for the derivative dx_k/dt at time $t = t_0$. Solving this equation, we find dx_k/dt for $t = t_0$ and can thus avoid the singularity at $t = t_0$.

In this manner we can study the behavior of isotherms near singularities which may arise at their intersection with the boundaries of the region. This topic is taken up in more detail for specific examples in section 5.

Other cases of the behavior of isotherms near the boundaries $x = g_1(t)$ and $x = g_2(t)$ can be discussed in a completely analogous manner. More complicated boundary conditions than (3.1) can be considered, e.g., boundary conditions at unknown boundaries.

The initial condition for Eq. (1.3) is usually specified in the form $u = h(x)$ at $t = 0$, where h is a specified function. In order to obtain the initial conditions for isotherms $x = x_k(t)$, we find the roots of the equations $h(x) = u_k$ for $k = 0, 1, \ldots, N$. If x_k, x_k', x_k'', etc. are the roots of this equation for some fixed k, these roots are also the initial values $x_k(0)$, $x_k'(0)$, $x_k''(0)$, etc. for the isotherms corresponding to the u_k value.

The number of isotherms beginning at $t = 0$ and corresponding to the value $u = u_k$ is equal to the number of roots of the equation $h(x) = u_k$. If the function h(x) is discontinuous, several isotherms may emerge from its discontinuity, corresponding to those u_k values enclosed between the limiting values of the function h(x) on the left and right of the discontinuity.

4. Discussion of the method and its generalization. As was shown above, solution of the boundary-value problem for nonlinear partial differential equation (2.2) with the discontinuous function F(u) in the form (2.1) leads to the solution of the nonlinear system of ordinary differential equations for the $x = x_k(t)$ isotherms corresponding to the discontinuities $u = u_k$ in the function $F(u)$. We have shown how to construct these equations in various cases for interior and boundary isotherms [Eqs. (2.5)–(2.7), (3.2)].

The isotherms may end (or disappear) either within the region or at its boundaries. New isotherms may begin at the boundaries, and in this case we can specify initial conditions for them. The indeterminacy arising in the isotherm equations at their intersection with the boundaries can be handled. Finally, we note that the boundary isotherms may convert into interior isotherms, and vice versa. Accordingly, the problem as formulated reduces to the Cauchy problem for a system of ordinary differential equations whose form and degree change with the time according to definite rules. This system can be easily integrated by familiar numerical methods, e.g., the Runge-Kutta method. After the problem is solved and all the isotherms $x_k(t)$ are determined, the solution $u(x, t)$ is found as a piecewise-linear function of x taking on the values $u = u_k$ for $x = x_k(t)$.

In principle, this method yields an exact solution of Eq. (2.2), so the question of the convergence of the method reduces to the following question: in what sense and under what conditions does the solution of Eq. (2.2) approximate the solution of Eq. (1.3) if the piecewise-linear function $F(u)$ is approximately equal to the function $f(u)$ in the sense of metric C, i.e., in terms of the maximum modulus of the difference?

It is obvious from physical considerations that when $F(u) \to f(u)$ the solution of Eq. (2.2) tends toward Eq. (1.3) (except, perhaps, for certain particular cases). This follows from the fact that two media having similar properties must behave similarly. Nevertheless, a rigorous mathematical demonstration of the convergence is desirable.

With essentially no changes, this method is also applicable for problems of cylindrical and spherical symmetry. In these cases Eq. (1.3) becomes

$$\frac{\partial f(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\nu}{x}\frac{\partial u}{\partial x}, \tag{4.1}$$

where $\nu = 1, 2$ for cylindrical and spherical symmetry, respectively, and x is the distance from the axis or center of symmetry. In this case, between the isotherms the solution $u(x, t)$ will satisfy the linear equation obtained by equating the right-hand side of Eq. (4.1) to zero. Solving this equation, we find

$$u = c_1 + c_2 \ln x \quad (\nu = 1),\; u = c_1 + c_2 x^{-1} \quad (\nu = 2). \tag{4.2}$$

The arbitrary constants c_1 and c_2 are determined from the conditions at the isotherms (i.e., conditions of the type $u = u_k$ at $x = x_k$). Evaluating these constants for the case treated in section 2, we find, instead of the first of relations (2.3), the following:

$$u = u_k + \frac{(u^- - u_k)\ln(x/x_k)}{\ln(x^-/x_k)} \quad (\nu = 1),$$
$$u = u_k + \frac{(u^- - u_k)(1/x - 1/x_k)}{1/x^- - 1/x_k} \quad (\nu = 2). \tag{4.3}$$

Analogous equations are obtained to replace the second of relations (2.3). Multiplying Eq. (4.1) by x^ν and integrating it from a to b, we find

$$x_k^\nu \frac{dx_k}{dt}(F^- - F^+) = \left(x^\nu \frac{\partial u}{\partial x}\right)\Big|_a^b.$$

Evaluating the right-hand side of this equation with the help of Eqs. (4.3), to replace Eqs. (2.3) here, we find

$$x_k \frac{dx_k}{dt}(F^- - F^+) = \frac{u^+ - u_k}{\ln(x^+/x_k)} - \frac{u^- - u_k}{\ln(x^-/x_k)},$$
$$x_k^2 \frac{dx_k}{dt}(F^- - F^+) = \frac{u^+ - u_k}{1/x_k - 1/x^+} - \frac{u^- - u_k}{1/x_k - 1/x^-} \tag{4.4}$$

for the cases $\nu = 1$ and $\nu = 2$, respectively. Equations (4.4) replace Eqs. (2.4) here. The remaining equations of sections 2–3 are changed in an analogous manner, but the discussion and method of solution remain the same.

We offer yet another interpretation of this method. In Eq. (1.3) we assume that $f(u)$ is differentiable and that u(x,

t) is a monotonic function of x. Using the variable replacements

$$\tau = t, \quad y = u(x,\ t), \quad x = X(y,\ \tau)\ , \tag{4.5}$$

we seek X as a function of the arguments y and τ. For the partial derivatives we find

$$u_t = -X_\tau / X_y, \quad u_x = 1 / X_y, \quad u_{xx} = X_y^{-1} (1 / X_y)_y\ .$$

In terms of these new variables, Eq. (1.3) becomes

$$-f'(y) X_\tau = (1 / X_y)_y\ .$$

If the derivatives with respect to y in this equation are replaced by piecewise-difference relations, i.e., if we use the straight-line method, we find equations of the form (2.4). Accordingly, this method may be considered a version of the straightline method for the equation in terms of the new variables in (4.5).

We take note of certain features and advantages of this method. Since the method converges to the Cauchy problem for a system of ordinary differential equations, it is simply solved on a computer. The method can be used to solve the problem for arbitrary nonlinear functions $f(u)$ having any number of discontinuities.

The calculation of the phase-transition fronts is carried out automatically since they are included in the number of isotherms and do not present any difficulties. The choice of the piecewise-constant function F(u) approximating $f(u)$ is largely arbitrary; this circumstance can be used to improve the accuracy.

If the function $f(u)$ increases slowly in one u range and rapidly in another, the points u_k must be distributed at greater intervals in the first range and at smaller intervals in the second so that the increments in the function F(u) will be approximately the same. If the solution u(x, t) has sharp gradients in certain regions of the x, t plane, this will automatically be taken into account by the approach of the isotherms and will not lead to any loss in accuracy.

This method can be conveniently used to solve the optimum-control problem for nonlinear thermal processes, since such problems can be immediately reduced to variational problems for systems described by ordinary differential equations.

This method, which can easily take into account any nonlinearities and phase transitions, therefore has certain advantages over other known methods (e.g., the finite-difference methods) in certain cases. The method may be useful for solving nonlinear heat-conduction problems, the Stefan problems, and other physical and mechanical problems described by a nonlinear equation of the parabolic type. The method can be extended to problems involving several spatial variables, but the calculation procedure is significantly complicated thereby.

We note yet another generalization of the method, which was pointed out to the author by L. A. Chudov. We consider a system of partial differential equations

$$\frac{\partial v_i}{\partial t} = L_i(v), \quad v = (v_1, \ldots v_n), \ i = 1, \ldots n\ , \tag{4.6}$$

where v(t, x) is the unknown vector function and L_i are arbitrary nonlinear differential operators containing derivatives of v of arbitrary order with respect to the scalar spatial variable x.

The operators L_i may depend explicitly on x and t. We let $x_{ij}(t)$ be the level line of the function $v_i(x, t)$ on which it assumes the constant value $v_i(x, t) = v_{ij}$. Then on this line we have

$$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x}\frac{dx_{ij}}{dt} = 0 \quad (i = 1,\ 2 \ldots n;\ j = 1,\ 2 \ldots)\,.$$

From this, and using (4.6), we find an equation for the level lines:

$$\frac{dx_{ij}}{dt} = \frac{\partial v_i / \partial t}{\partial v_i / \partial x} = \frac{L_i(v)}{\partial v_i / \partial x} \quad (i = 1,\ 2 \ldots n;\ j = 1,\ 2 \ldots)\,. \tag{4.7}$$

The right-hand side of Eqs. (4.7) can be replaced by various finite-difference approximations employing the selected grid of v_{ij} values and $x = x_{ij}(t)$ values at the level line. System (4.7) then becomes analogous to Eqs. (2.5).

5. Examples. Here we consider numerical solutions of the same boundary-value problem for two media with $f(u)$ functions

$$f_1(u) = u, \quad f_2(u) = \begin{cases} 0.75\,u & \text{for } u < 0.8 \\ 0.4 + 0.5\,u + 0.2\,u^2 & \text{for } 0.8 \leqslant u < 1.5 \\ u + 0.4\,u^2 & \text{for } u \geqslant 1.5\,. \end{cases} \tag{5.1}$$

These functions are plotted in Fig. 3.

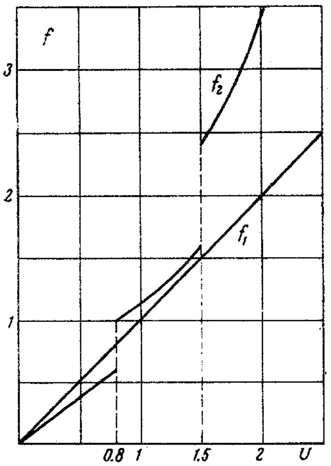

Fig. 3

The first case corresponds to the ordinary linear heat-conduction equation, while the second corresponds to a medium having two phase transitions. The boundary and initial conditions for both cases are

$$\partial u / \partial x = 0 \text{ for } x = 0, \; u = 2t + 1 \text{ for } x = 1, \; u = x^2 \text{ for } t = 0\,. \tag{5.2}$$

The points u_k in Eq. (2.1) are chosen to be equidistant; i. e., we choose $u_k = kH$, where $H = 0.05$. In the region $t \geq 0$, $0 \leq x \leq 1$ in which we seek the solution, the function $u(x, t)$ is a monotonic function of both x and t in these examples, so the isotherms are arranged in the x, t plane in a manner such that $x_k(t) < x_l(t)$ for $k < l$. As t increases, the isotherms move to the left; new isotherms arise at the right ($x = 1$) boundary, while isotherms end at the left ($x = 0$) boundary (Fig. 4). We assume that at some time t $x_k(t)$ isotherms with indices $i \leq k \leq j$ lie in the interval $0 < x < 1$; here i indicates the isotherm at the extreme left, while j indicates that of the extreme right. Taking into account the general equalities (2.5), (3.2), and (3.3) and condition (5.2), we write the isotherm equations as follows:

$$\frac{dx_i}{dt} = \frac{H}{(F_{i-1} - F_i)(x_{i+1} - x_i)} < 0\,,$$

$$\frac{dx_k}{dt} = \frac{H}{F_k - F_{k-1}} \left(\frac{1}{x_k - x_{k-1}} - \frac{1}{x_{k+1} - x_k} \right) \quad (i < k < j)\,,$$

$$\frac{dx_j}{dt} = \frac{H}{F_j - F_{j-1}} \left[\frac{1}{x_j - x_{j-1}} - \frac{(2t+1)/H - j}{1 - x_j} \right]. \tag{5.3}$$

To obtain the initial conditions we write the equation

$$u(x, 0) = x^2 = u_k = kH \qquad (H = 0.05)\,.$$

Then we find the initial conditions for those isotherms which begin on the [0, 1] interval of the x-axis:

$$x_k(0) = \sqrt{kH}, \qquad H = 0.05, \; i \leqslant k \leqslant j, \; i = 1, \; j = 20\,. \tag{5.4}$$

Isotherms with $j > 20$ begin at the line $x = 1$ at a time t given by $u(1, t) = 2t + 1 = jH$; i. e., for them we have

$$x_j(t_j) = 1, \; t_j = 1/2\,(jH - 1)\,. \tag{5.5}$$

When a new isotherm arises, it becomes the bounding isotherm on the right, i. e., the j-th isotherm in the notation of (5.3). The last equation in (5.3) has a singularity when (5.5) holds. Resolving the indeterminant form in (5.3) (see section 3), we find at $t = t_j$ that

$$z = a_1 + a_2 / z\ , \tag{5.6}$$

where

$$z = \frac{dx_j(t_j)}{dt},\quad a_1 = \frac{H}{(F_j - F_{j-1})[1 - x_{j-1}(t_j)]} > 0\ ,$$
$$a_2 = 2/(F_j - F_{j-1}) > 0\ . \tag{5.7}$$

Of the two roots of the quadratic equations (5.6), we select the negative one:

$$z = \frac{dx_j(t_j)}{dt} = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2} < 0\ , \tag{5.8}$$

which describes the motion of the isotherm on the left of the x = 1 boundary in the x, t plane. Equations (5.7) and (5.8) describe the asymptotic behavior of the $x_j(t)$ isotherm upon its appearance, i.e., under condition (5.5).

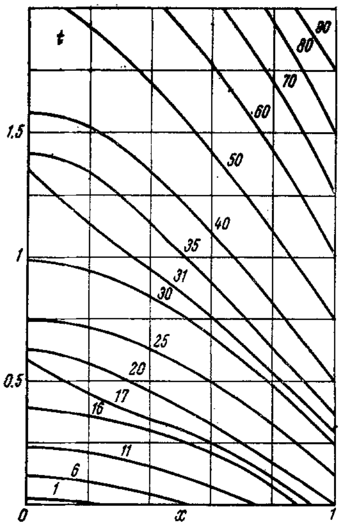

Fig. 4

It can be shown that the point $t = t_j$, $x_j = 1$ is a saddle-point singularity for system (5.3), and that only one isotherm leaves this point along the direction (5.8). There are no other singularities of system (5.3). The disappearance of the isotherms in these examples occurs at the x = 0 boundary when the decreasing function $x_i(t)$ vanishes. Then the role of the i-th, i.e., bounding isotherm on the left in Eqs. (5.3), is played by the isotherm x_{i+1}.

The system of isotherm equations (5.3) has been integrated numerically on a computer by the Runge-Kutta method with a constant step of $5 \cdot 10^{-4}$ for the first example in (5.1) and $2.5 \cdot 10^{-4}$ for the second example. The initial conditions and the i and the j values for t = 0 were specified in the form (5.4).

At each step of the integration, the condition for the disappearance of the i-th isotherm was checked (the condition $x_i \leq 0$), as was the condition for the appearance of a new (j + 1)-th isotherm (the condition $t \geqslant t_{j+1} = 1/2\,[(j + 1)\,H - 1]$). When the first condition holds, the i value increases by unity; when the second condition holds, the j value increases by unity. When a new isotherm appears, its behavior near the singularity is specified by the asymptotic behavior above. After the isotherm calculation, the solution u(x, t) can be determined by a linear interpolation over x between neighboring isotherms.

In the first of these examples [see (5.1)], with $f_1(u) = u$, there is an exact solution of boundary-value problem (5.2) for Eq. (1.3). This solution is $u(x, t) = x^2 + 2t$, and the isotherms in the x, t plane here are parabolas. The existence of a simple exact solution permits an estimate of the error in this method. We show for the first example the numerical solution u^0 and the exact solution $u = x^2 + 2t$:

	$t=1$			$t=2$		
$x =$	0.0770	0.5004	0.8945	0.0799	0.5009	0.9747
$u^0 =$	2.0000	2.2500	2.8000	4.0000	4.2500	4.9500
$u =$	2.0059	2.2504	2.8001	4.0064	4.2509	4.9500

To eliminate the interpolation error, we chose the arguments x, t here to lie on one of the isotherms plotted numerically. The numerical solutions were found from $u^0 = kH = 0.05\,k$, where k denotes the isotherm on which the point is taken. As this table shows, the agreement between the numerical solution and the exact solution is good. The isotherms plotted here also turn out to be close to the exact isotherms (parabolas).

The second example corresponds to the nonlinear discontinuous function $f_2(u)$ from (5.1) (Fig. 3). Figure 4 shows some isotherms obtained by a numerical solution for this example. The curve labels indicate the isotherms; on the k-th isotherm, we have, as before, $u = kH = 0.05\,k$. The 16-th and 30-th isotherms, at which $u = 0.8$ and $u = 1.5$, are, according to (5.1), phase-transition fronts (the function $f_2(u)$ has a discontinuity here). These isotherms separate states of the medium having different properties, as is clearly shown by the behavior of the isotherm field in Fig. 4.

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